Intersection patterns of tubes in Euclidean space and an oscillatory integral operator

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1 Outline

1. Tube doubling problem and conjecture.
2. Kakeya conjecture on tubes.
3. Besicovitch's example in $\mathbb{R}^2$ and its connection with the above conjectures.
4. L^p estimates for an oscillatory integral operator.

2 Tube doubling problem

Motivation. The ball doubling problem.
Notations.

1. $T_i \subset \mathbb{R}^n$ are cylindrical tubes of radius 1 and length N.
2. $2T_i$ is the concentric tube of radius 2 and length $2N$ formed by dilating T_i around its center by a factor of 2.

Sometimes notation 1 will be abbreviated as "a $1 \times N$ tube".

Question 2.1. *Is there a constant C_n so that for any N, for any set of $1 \times N$ tubes T_i in $\mathbb{R}^n$,*

$$|\cup_i 2T_i| \leq C_n |\cup_i T_i|?$$

The answer is no. Besicovitch gave a counterexample, in which,

$$|\cup_i 2T_i| \gtrsim \frac{\log N}{\log \log N} |\cup_i T_i| .$$

Such examples lead to the following conjecture[1]:

[1] I will skip this conjecture

Conjecture 2.2 (Tube doubling conjecture). *For any dimension n, for any $\epsilon > 0$, there is a constant $C_n(\epsilon)$, so that the following estimate holds for any N. If T_i are tubes of radius 1 and length N, then*

$$|\cup_i 2T_i| \le C_n(\epsilon) N^\epsilon |\cup_i T_i|.$$

3 Kakeya conjecture on tubes

Notations.

1. For a tube $T \subset \mathbb{R}^n$ as above, we write $v(T) \in S^{n-1}$ for a unit vector parallel to the axis of symmetry of T.

There are two choices of $v(T)$, differing by a sign. We call $v(T)$ the direction of the tube T.

Definition 3.1. *Suppose that $T_i \subset \mathbb{R}^n$ are tubes of radius 1 and length N. $\{T_i\}$ is a Kakeya set of tubes if $\{v(T_i)\}$ is $\frac{1}{N}$-separated and $\frac{2}{N}$-dense in S^{n-1}.*

$\frac{1}{N}$-separation and $\frac{2}{N}$-dense ensure that $\#\{T_i\} \sim N^{n-1}$.

Question 3.2. *Let $\{T_i\}$ be a Kakeya set of tubes. How small can $|\cup_i T_i|$ be?*

A trivial bound is $\#\{T_i\} \times |T_i| \sim N^{n-1}N = N^n$.

The construction of Besicovitch mentioned above gives a Kakeya set of tubes in the plane with

$$|\cup_i T_i| \lesssim \frac{\log\log N}{\log N} N^2.$$

This example leads to the following conjecture.

Conjecture 3.3 (Kakeya conjecture, tube version). *In any dimension $n \ge 2$, for any $\epsilon > 0$, there is a constant $C_{n,\epsilon}$ so that for any N the following holds. For any Kakeya set of tubes $T_i \subset \mathbb{R}^n$ of radius 1 and length N,*

$$|\cup_i T_i| \ge C_{n,\epsilon} N^{n-\epsilon}.$$

The conjecture is known for dimension 2 but is still open for all $n \ge 3$.

4 Example of Besicovitch in the plane

4.1 Construction and application to the Kakeya problem

Goal. Constructing a set of N rectangles in the plane, R_j, with width $\frac{1}{N}$ and length 1, with slopes changing evenly between 0 and 1, and with a lot of overlap.
Construction.

Step 1. For integers $0 \leq j \leq N-1$, let $\ell_j \colon [0,1] \to \mathbb{R}$ be a list of affine linear functions of the form

$$\ell_j(x) = \frac{j}{N}x + H(j).$$

Step 2. Let R_j be the $\frac{1}{N}$ neighbourhood of the graph of ℓ_j, which contains a rectangle of width $\frac{1}{N}$ and length 1.

Step 3. Choose $H(j)$ so that the following theorem holds true.

Theorem 4.1. *Suppose that N is an integer of the form A^A for some large integer A. Let R_j be defined as above. If we choose the constants $H(j)$ correctly, then*

$$|\cup_j R_j| \lesssim A^{-1} \lesssim \frac{\log\log N}{\log N}.$$

The proof of Theorem 4.1 needs some careful analysis on the slope of ℓ_j. More precisely, we need to consider different scales. We expand j/N in base A:

$$\frac{j}{N} = \sum_{a=1}^{A} j(a)A^{-a},$$

where $j(a)$ are the digits in the base A decimal expansion of j/N. The different values of a represent different scales. We will choose $H(j)$ so that the following key estimate holds.

Proposition 4.2. *Suppose $1 \leq b \leq A$. If $j(a) = j'(a)$ for $1 \leq a \leq b-1$, then for all $x \in [\frac{A-b}{A}, \frac{A-b+1}{A}]$,*

$$|\ell_j(x) - \ell_{j'}(x)| \leq 4A^{-b}.$$

Proof of Theorem 4.1. Assume Proposition 4.2 holds. Then for each $1 \leq b \leq A$, we focus on the

$$S_b := (\cup_j R_j) \cap \left(\left[\frac{A-b}{A}, \frac{A-b+1}{A} \right] \times \mathbb{R} \right).$$

For the given choice of $j(a)$ with $1 \leq a \leq b-1$, we further divide S_b into

$$S_b^{j(1),\ldots,j(b-1)} := (\cup_{j'} R_{j'}) \cap \left(\left[\frac{A-b}{A}, \frac{A-b+1}{A} \right] \times \mathbb{R} \right),$$

where the union is take over all j' satisfying $j'(a) = j(a)$ for $1 \leq a \leq b-1$. We apply Proposition 4.2 to $S_b^{j(1),\ldots,j(b-1)}$ and take the summations over all possible values of $j(1), \ldots, j(b-1)$ and then over b. $\square$

It remains to choose $H(j)$. We write it as a sum of A different terms with different orders of magnitude, more precisely, we write

$$H(j) = \sum_{a=1}^{A} h(a)j(a)A^{-a},$$

where $h(a) \in [-1,1]$ is a constant that we can choose later. It will be shown that we can take $h(a) := -\frac{A-a}{a}$ so that one more scale can be cancelled out.

4.2 Application to the tube doubling problem

Recall that $v(R_j)$ is only defined up to sign, and we can make the choice so that the x-component of $v(R_j)$ is negative.
Notations.

1. Define R_j^+ to be the translation of R_j by $10v(R_j)$.

Observations.

1. R_j^+ is contained in $100R_j$.

2. R_j^+ are disjoint.

As a result, we get

$$|\cup_j 100R_j| \geq \left|\cup_j R_j^+\right| = \sum_j |R_j| \gtrsim \frac{\log N}{\log\log N} |\cup_j R_j|.$$

Therefore, this example can also be used as a slightly weaker counterexample to the tube doubling problem.

5 An oscillatory integral operator

We study L^p estimates for the operator T_α defined by

$$T_\alpha f := f * K_\alpha = \int_{\mathbb{R}^n} f(y) K_\alpha(\cdot - y)\,\mathrm{d}y,$$

where

$$K_\alpha(x) := (1+|x|)^{-\alpha}\cos|x|.$$

Properties of K_α.

1. K_α is radial.

2. K_α oscillates with the radius because of the function $\cos|x|$. More precisely, the kernel K_α has positive and negative parts, and so in the convolution $f * K_\alpha$, some cancellation can occur.

We will focus on estimates of the form $\|T_\alpha f\|_p \lesssim \|f\|_p$ with the assumption that $0 < \alpha < n$.

5.1 Behaviour of T_α on some spherically symmetric examples

1. $f_1 := \chi_{B_r}$, where r is small. As a result, $p > \frac{n}{\alpha}$.

2. $f_2 = \chi_{B_r}\,\mathrm{Sign}\,(\cos|x|)$ where r is large. Consequently, $p \leq \frac{n}{n-\alpha}$.

3. [2] $f_3 := \chi_{B_r} K_{n-\alpha}$ where r is large. As a result, $p \neq \frac{n}{n-\alpha}$.

[2] I may skip this example

5.2 Example related to the long thin tube

Now, we consider an oscillating function supported on a long thin tube.
Notations.

1. Let T be a cylinder of length $L \gg 1$ and radius $(1/1000)L^{\frac{1}{2}}$.
2. Let v_T be a unit vector parallel to the axis of the cylinder.
3. T^+ denote the cylinder we get by translating T by $10Lv_T$.

The cylinder may point in any direction.
Example. Let

$$f_T(x) := \chi_T(x)e^{iv_T \cdot x}.$$

Proposition 5.1. *Fix a dimension $n \geq 2$. For all L sufficiently large, the following holds. If f_T and T^+ are defined as above, then for every $x \in T^+$ we have*

$$|T_\alpha f_T(x)| \geq L^{\frac{n+1}{2}-\alpha}.$$

Corollary 5.2. *If $\alpha < \frac{n+1}{2}$, then for every $p \in [1, \infty]$, as $L \to \infty$,*

$$\frac{\|T_\alpha f_T\|_p}{\|f_T\|_p} \to \infty.$$

Proposition 5.3. *If $\|T_\alpha f\|_p \lesssim \|f\|_p$ holds for f_1, f_2, f_3, and f_T, then*

$$\frac{n}{\alpha} < p < \frac{n}{n-\alpha} \quad \textit{and} \quad \alpha \geq \frac{n+1}{2}.$$

References

[Gut16] Larry Guth. *Polynomial Methods in Combinatorics*, volume 64 of *University Lecture Series*. American Mathematical Society, 2016.